

# On the Dynamical Interpretation for the Quantum-Measurement Projection Postulate

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## Abstract

An apparatus model with discrete momentum space suitable for the exact solution of the problem is considered. The special Hamiltonian of its interaction with the object system under consideration is chosen. In this simple case it is easy to illustrate how difficulties in constructing the dynamical interpretation of selective collapse could be overcome without any limiting procedure. For this purpose one can apply either averaging with respect to a non-quantum parameter or reducing the algebra of joint-system operators (i. e. passing from algebra  $\mathcal{A}$  of operators to a subalgebra  $\mathcal{A}_0$ ). The latter procedure implies averaging with respect to apparatus quantum variables not belonging to  $\mathcal{A}_0$ .

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# 1 Introduction

In this paper we consider the dynamical interpretation of the selective collapse in the one dimensional case, when momentum of the apparatus has discrete spectrum of eigenvalues. This simplifies the problem of dynamical corroboration of the von Neumann projection postulate. The idea to consider the case, when one of two main conjugate dynamic variables (momentum or coordinate) are discrete, and to take the apparatus state commuting with discrete variable belongs to one of the authors [1].

The approach to the problem of the selective collapse interpretation is quite ordinary and is well known from the time of von Neumann [2]. The collapse of the quantum object state, which takes place during measurement of an object variable  $X$  with discrete spectrum, is interpreted with the help of interaction between object  $S$  and apparatus  $A$  (the latter being in the quasiclassical state) and with the help of the subsequent classical-like measurement of some apparatus variable  $Y$ . In our case  $Y$  depends on quantum momentum  $\hat{p}$ . Moreover, in this case the evolution operator can exactly realize the transformation of the product wave function

$$|\varphi\rangle \otimes |y_0\rangle = \sum_j c_j |x_j\rangle \otimes |y_0\rangle$$

for the joint system into the correlated one

$$\sum_j c_j |x_j\rangle \otimes |y_j\rangle.$$

This transformation was proposed by von Neumann for the measurement interpretation. Here  $|x_j\rangle$  are the eigenfunctions of  $X$ , and  $|y_j\rangle$  are the eigenfunctions of  $Y$ . In contrast to the von Neumann theory, use of the mixed apparatus state or to be exact the quasi-classical state is desirable because eigenvalues  $\{y_j\}$  of  $Y$  can only be distinguished from one another macroscopically in a quasi-classical state, the appropriate measured operator  $Y$  being chosen. Moreover we use an averaging procedure of the apparatus state (see [3]). This procedure helps to overcome the difficulties connected with the dynamic interpretation of the collapse making the apparatus state compatible with  $Y$ .

Our goal is to interpret the selective collapse

$$\rho_S \rightarrow \frac{1}{w_l} E_l \rho_S E_l \quad (1.1)$$

( $w_l = \text{Tr}_S \rho_S E_l$ ) of the density matrix of the quantum object S. According to the projection postulate it takes place when the result  $x_l$  of measurement of the operator  $X = \sum_j x_j E_j$  becomes known. Here  $E_j$  are the orthogonal projectors ( $E_j E_k = E_j \delta_{jk}$ ,  $\sum_j E_j = I_S$ ).

Our treatment is restricted to the following assumptions.

(i) The coordinate space of the apparatus model is finite, namely it is of the length  $L$  and is curved into itself (like circumference), the coordinate spectrum being, say, the interval  $[-L/2, L/2]$ . This means that the shift  $q \rightarrow q + a$  gives  $q + a - L$  if  $L/2 < q + a < 3L/2$  and  $q + a + L$  if  $-L/2 > q + a > -3L/2$ . The pointer on a fixed axis (for it  $q = \varphi$  is the angle,  $L = 2\pi$ ) or a box with periodic boundary conditions may serve as examples. For an arbitrary  $L$  the apparatus momentum has eigenvalues  $p_k = 2\pi\hbar k/L$ .

(ii) The initial apparatus density matrix  $\rho_A$  of the apparatus is compatible with momentum  $\hat{p}$ , i. e. is diagonal in the momentum representation

$$\rho_{kl} := \langle p_k | \rho_A | p_l \rangle = w_k^0 \delta_{kl}. \quad (1.2)$$

Besides we suppose that

$$w_k^0 = 0 \quad \text{at} \quad |k| > m. \quad (1.3)$$

This compatible density matrix is only possible because of discrete character of the momentum spectrum. In fact, its continuous variant

$$\langle p' | \rho_A | p \rangle = w^0(p) \delta(p' - p)$$

is impossible because this operator has infinite trace (if  $w^0(p)$  is not equal to zero everywhere).

(iii) The interaction Hamiltonian is of the form

$$\mathcal{H}_{\text{int}}(t) = -B \otimes (\gamma \hat{q} + \lambda I_A) \delta(t), \quad (1.4)$$

where  $\gamma, \lambda$  are interaction constants,  $I_A$  is the apparatus identity operator. Of course, the presence of delta-function on the right-hand side of (1.4) makes the process of interaction somewhat unrealistic. This delta-function type of interaction was applied in [4] in the recurrent variant for realizing continuous observation.

The operator

$$B = f(X) = \sum_j b_j E_j = \sum_j f(x_j) E_j \quad (1.5)$$

enters the right-hand side of (1.4), the function  $f$  being chosen in such a way that all eigenvalues  $b_j$  of  $B$  be multiple to the same quantity  $a > 0$ :

$$b_j = n_j a. \quad (1.6)$$

Here  $n_j$  are integers that increase with increasing  $j$ . Transformation  $b_j = f(x_j)$  is supposed to be non-degenerate. The necessity of (1.6) will be clear later.

To obtain the collapse (1.1) of the object system state, the measurement of the variable  $Y$  depending on the apparatus momentum will be made. The matrix density (1.2) is very convenient for measuring  $Y$  because it commutes with  $p$  and therefore with  $Y(\hat{p})$ .

In the general case the selective quantum collapse

$$\rho_A \rightarrow \frac{1}{w'_l} P_l \rho_A P_l \quad (1.7)$$

of the apparatus state takes place after measurement of  $Y = \sum_j y_j P_j$  if the measurement result  $y_l$  becomes known. Here  $P_j$  are eigen-orthoprojectors of  $Y$  ( $\sum_j P_j = I_A$ ) and  $w'_l = \text{Tr}_A P_l \rho_A$ . Averaging  $\sum_l w'_l \tilde{\rho}_l$  *a posteriori* matrix densities

$$\tilde{\rho}_l = \frac{1}{w'_l} P_l \rho_A P_l$$

does not give *a priori* matrix  $\rho_A$  in the general case. This means that the condition of consistency

$$\sum_l w'_l \tilde{\rho}_l = \rho_A \quad \text{or} \quad \sum_l P_l \rho_A P_l = \rho_A \quad (1.8)$$

is not obliged to be met. In our case the projectors  $P_j$  defined by

$$P_j = \vartheta_j(\hat{p}) \quad (1.9)$$

commute with  $\rho_A$  and therefore the consistency condition (1.8) is met. The function  $\vartheta_j(\xi)$  is defined by (4.6).

As was pointed out in [3], the quasi-classical collapse

$$\rho_A \rightarrow \frac{1}{w'_l} \rho_A * P_l \quad (1.10)$$

obviously meeting the consistency condition can be applied in some cases. Here the operation  $*$  is defined with the help of the Wigner transformation (2.7), (2.8) denoted by  $\mathcal{W}$ . To be exact, in our case

$$A * B = L \mathcal{W}^{-1} \{ \mathcal{W}[A] \mathcal{W}[B] \}.$$

For projectors (1.9) we have

$$L \mathcal{W}[\vartheta_l(\hat{p})] = \vartheta_l(p_j),$$

and (1.10) is equivalent to

$$\rho_A \rightarrow (w'_l)^{-1} \mathcal{W}^{-1} [\mathcal{W}[\rho_A] \vartheta_l(p_j)]$$

or if we apply  $\mathcal{W}$  to both sides of the formula

$$w(q, p_j) \rightarrow \frac{1}{w'_l} w(q, p_j) \vartheta_l(p_j) \quad (1.11)$$

This is nothing else as transition to the conditional distribution, which is well-known non-quantum procedure. Using (2.10), one can easily see that collapse (1.10), (1.11) is exactly equivalent to (1.7) in our simple case. Because of this fact and because the condition (1.8) is met in our case, we call the measurement of  $Y = \sum y_k \vartheta_k(p)$  classical-like.

## 2 The initial apparatus state in other representations

Eigenfunctions of momentum  $\hat{p}$  corresponding to the eigenvalues  $p_k = 2\pi\hbar k/L$  are

$$\psi_k(q) = L^{-1/2} \exp(ip_k q/\hbar). \quad (2.1)$$

(the coordinate representation). Using expression on the right-hand side taken at various  $k$  we readily write down the matrix elements

$$V_{qk} = L^{-1/2} \exp(2\pi i k q/L) \quad (2.2)$$

of the unitary operator  $V$  that transforms  $\hat{p}$ -representation to  $\hat{q}$ -representation and vice versa. Thus  $\hat{q}$ -representation of the density matrix is

$$\rho(q', q) := \langle q' | \rho_A | q \rangle = \sum_{kl} V_{q'k} \rho_{kl} V_{lq}^\dagger,$$

or due to (1.2) and (2.2)

$$\rho(q', q) = L^{-1} \sum_k \exp[2\pi i(q' - q)k/L] w_k^0 \quad (2.3)$$

Therefore the coordinate probability density  $w^0(q) = \rho(q, q)$  is uniform

$$w^0(q) = 1/L.$$

Hence we find the coordinate mean  $\langle q \rangle = 0$  and mean square

$$\sigma_q^2 := \langle q^2 \rangle = \frac{1}{L} \int_{-L/2}^{L/2} q^2 dq = \frac{L^2}{12}. \quad (2.4)$$

On the other hand the momentum mean square is

$$\sigma_p^2 = \sum_{k=-m}^m p_k^2 w_k^0 = 8\pi^2 \hbar^2 L^{-2} \sum_{k=1}^m k^2 w_k^0 \quad (2.5)$$

if  $w_{-k}^0 = w_k^0$ . According to (2.4), (2.5) we have

$$\sigma_q^2 \sigma_p^2 = \frac{2}{3} \pi^2 \hbar^2 \sum_{k=1}^m k^2 w_k^0. \quad (2.6)$$

It should be noted that we get  $\sigma_q \sigma_p = 0$  from (2.6) if  $m = 0$ , i.e. if  $w_k^0 = \delta_{k0}$ . This equation is very unusual since it violates the Heisenberg uncertainty relation  $\sigma_q \sigma_p \geq \hbar/2$ . Possibility of this paradox is argued in Appendix.

When  $\sigma_q \sigma_p \gg \hbar$ , the apparatus is in a quasi-classical state. We will suppose that this inequality is valid because the direct macroscopic observation of a physical quantity is possible only in this case. Owing to (1.3) and normalization condition  $\sum_k w_k^0 = 1$ , the inequality  $m \gg 1$  is a necessary condition for  $\sigma_q \sigma_p \gg \hbar$ . For many distributions, i.e. for the uniform one formula  $m \gg 1$  is also a sufficient condition of a quasi-classical state.

Another representation of the apparatus state is the Wigner distribution, which in our case takes the form

$$\begin{aligned} w(q, p_j) &= \frac{1}{L} \int_{-L/2}^{L/2} \exp\left(-\frac{i}{\hbar} u p_j\right) \rho\left(q + \frac{u}{2}, q - \frac{u}{2}\right) du \\ &= \frac{1}{L} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \exp\left[\frac{i}{\hbar} q(p_k - p_l)\right] \Delta\left(\frac{k+l}{2} - j\right) \rho_{kl}. \end{aligned} \quad (2.7)$$

Here  $\Delta(\eta) = \int_{-1/2}^{1/2} \exp(2\pi i \eta v) dv$ , i. e.

$$\Delta(\eta) = \frac{\sin(\pi\eta)}{\pi\eta} = \begin{cases} \delta_n & \text{at } \eta = n, \\ (-1)^n \pi^{-1} / (n + \frac{1}{2}) & \text{at } \eta = n + \frac{1}{2} \end{cases}$$

( $n$  is integer). We denote transformation (2.7) by  $\mathcal{W}$ ;

$$\mathcal{W}[\rho_A] = w(q, p_j). \quad (2.8)$$

It is easy to check that  $w(q, p_j)$  has properties

$$\sum_j w(q, p_j) = \rho(q, q), \quad \int w(q, p_j) dq = \rho_{jj} = w_j^0$$

usual for the Wigner distribution. Moreover the formula

$$\text{Tr}_A G \rho_A = L \sum_{j=-\infty}^{\infty} \int_{-L/2}^{L/2} \mathcal{W}[G] \mathcal{W}[\rho_A] dq \quad (2.9)$$

is valid. For the special matrix density (1.2) we get

$$w(q, p_j) = w_j^0 / L. \quad (2.10)$$

### 3 Interaction between the object system S and apparatus

Let  $H_S$  be a Hamiltonian acting in the Hilbert space  $\mathcal{H}_S$  of the object system S. The apparatus Hamiltonian  $H_A$  is an operator acting in  $\mathcal{H}_A$ . Interaction between S and A that lasts very short time from  $t = -\varepsilon$  to  $t = \varepsilon$  is described by the interaction Hamiltonian (1.4) acting on  $\mathcal{H}_S \otimes \mathcal{H}_A$ ,  $B$  being the S-system operator with discrete eigenvalues (1.6). Its measurement or — what is equivalent — measurement of  $X$  is to be interpreted. Hence the total Hamiltonian assumes the form

$$H(t) = H_S \otimes I_A + I_S \otimes H_A - \gamma B \otimes q \delta(t) - \lambda B \otimes I_A \delta(t). \quad (3.1)$$

The state of the joint system S + A at the initial instant  $t_0 = -\varepsilon$  is given by the density matrix

$$\rho(-\varepsilon) = \rho_S \otimes \rho_A. \quad (3.2)$$

In the Schrödinger picture the density matrix depends on time as

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0), \quad (3.3)$$

where the evolution operator  $U$  is given by

$$U(t, t_0) = \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t_1} H(t) dt \right]. \quad (3.4)$$

Here  $\mathcal{T}$  denotes the time ordering of operators  $H(t)$ , namely the greater  $t$  is the more to the left  $H(t)$  stands. We choose  $t_1 = \varepsilon > 0$ , where  $\varepsilon$  is a very small number. Then (3.3) gives

$$\rho(\varepsilon) = \exp \left[ \frac{i}{\hbar} B \otimes (\gamma \hat{q} + \lambda I_A) \right] (\rho_S \otimes \rho_A) \exp \left[ -\frac{i}{\hbar} B \otimes (\gamma \hat{q} + \lambda I_A) \right] \quad (3.5)$$

owing (3.1), (3.2), (3.4). We will use the orthogonal projectors  $\{E_j\}$  corresponding to the operator  $B = \sum_j b_j E_j$ . As is well known, for them

$$I_S = \sum_j E_j. \quad (3.6)$$

By virtue of (3.6) we can take  $\sum_i E_i \rho_S \sum_j E_j$  instead of  $\rho_S$  in (3.5) and obtain

$$\rho(\varepsilon) = \sum_{ij} \exp \left[ \frac{i}{\hbar} B \otimes (\gamma \hat{q} + \lambda I_A) \right] (E_i \rho_S E_j \otimes \rho_A) \exp \left[ -\frac{i}{\hbar} B \otimes (\gamma \hat{q} + \lambda I_A) \right]. \quad (3.7)$$

But  $BE_i = b_i E_i$ ,  $E_j B = E_j b_j$  and  $g(B \otimes D)E_j \otimes \rho_A = E_j \otimes (g(b_j D)\rho_A)$  for an arbitrary  $c$ -function  $g$ . Therefore (3.7) yields

$$\rho(\varepsilon) = \sum_{ij} E_i \rho_S E_j \otimes \exp \left[ \frac{i}{\hbar} b_i (\gamma \hat{q} + I_A) \right] \rho_A \exp \left[ -\frac{i}{\hbar} b_j (\gamma \hat{q} + I_A) \right]. \quad (3.8)$$

Now we use formulas (1.4) and let

$$a\gamma = 2\pi\hbar(2m+1)/L.$$

Then in the apparatus coordinate representation

$$\langle q' | \rho(\varepsilon) | q \rangle = \sum_{ij} E_i \rho_S E_j e^{i(n_i - n_j)x} \exp \left[ 2\pi i (2m+1)(n_i q' - n_j q) L^{-1} \right] \rho(q', q)$$



with  $\chi = a\lambda/\hbar$ . Substituting (2.3) into the right-hand side and passing to the  $p$ -representation hence we get

$$\langle p_r | \rho(\varepsilon) | p_s \rangle = \sum_{ij} E_i \rho_S E_j e^{i(n_i - n_j)\chi} w_{r-(2m+1)n_i}^0 \delta_{r-s-(2m+1)(n_i - n_j)}. \quad (3.9)$$

The following Wigner transform follows from this result

$$\begin{aligned} \mathcal{W}[\rho(\varepsilon)]_{q,p_k} &= \sum_{ij} E_i \rho_S E_j e^{i(n_i - n_j)\chi} \exp \left[ 2\pi i (2m+1)(n_i - n_j) \frac{q}{L} \right] \\ &\quad \times w \left( q, p_k - \frac{1}{2}(p_{(2m+1)n_i} + p_{(2m+1)n_j}) \right), \end{aligned} \quad (3.10)$$

if all  $n_i + n_j$  are even.

## 4 The apparatus physical quantity that should be measured

Let us consider the expression

$$R(p_r) := \langle p_r | \rho(\varepsilon) | p_r \rangle, \quad (4.1)$$

which in our case, due to (3.9), assumes the form

$$R(p_r) = \sum_j E_j \rho_S E_j w_{r-(2m+1)n_j}^0. \quad (4.2)$$

It is an operator on  $\mathcal{H}_S$  and simultaneously the distribution of momentum  $p_j$ . We see that correlation exists between values  $b_j$  of  $B$  and those of  $p$ . In fact, the density matrix

$$\tilde{\rho}_j = \frac{1}{w_j} E_j \rho_S E_j, \quad (4.3)$$

in which  $B$  has definite value  $b_j$ , enters the same term  $w_j \tilde{\rho}_j w_{r-(2m+1)n_j}^0$  of the sum (4.2) as distribution  $w_{r-(2m+1)n_j}^0$  which is not 0 in the range

$$-2\pi\hbar m/L \leq p_r - 2\pi\hbar(2m+1)n_j/L \leq 2\pi\hbar m/L$$

(according to (1.3)) i.e.

$$2\pi\hbar[(2m+1)n_j - m]/L \leq p_r \leq 2\pi\hbar[(2m+1)n_j + m]/L. \quad (4.4)$$

Therefore determining the range, to which the momentum belongs, signifies determining the value of  $B$  and  $X$ . Let us denote the range (4.4) by  $S_j$ . Thus

$$w_{r-(2m+1)n_j}^0 = \begin{cases} w_{r-(2m+1)n_j}^0 & \text{at } p_r \in S_j, \\ 0 & \text{at } p_r \notin S_j. \end{cases} \quad (4.5)$$

Various ranges never overlap because  $n_{j+1} - n_j \geq 1$ . Let us take the enlarged not overlapping ranges  $\tilde{S}_j$  such that each  $\tilde{S}_j$  includes  $S_j$  and so that the sum  $\sum_j \tilde{S}_j$  is equal to the set of all  $p_j$ ,  $j = 0, \pm 1, \pm 2, \dots$ . This enlarging can be made in various ways. For example, we can take the points

$$s_j = 2\pi\hbar L^{-1} \left[ (2m+1) \frac{n_j + n_{j+1}}{2} \right]_{\text{IP}} \quad (4.6)$$

(the subscript IP means the integral part) lying approximately on the half-way between  $S_j$  and  $S_{j+1}$  and define  $\tilde{S}_j$  as the range  $s_{j-1} < p_k \leq s_j$ . Now we define the function

$$\vartheta_j(p_k) = \begin{cases} 1 & \text{at } p_k \in \tilde{S}_j, \\ 0 & \text{otherwise.} \end{cases} \quad (4.7)$$

From (4.5), (4.7) and since  $S_j$  is the subset of  $\tilde{S}_j$ , we have

$$w_{k-(2m+1)n_i}^0 \vartheta_j(p_k) = w_{k-(2m+1)n_i}^0 \delta_{ij}. \quad (4.8)$$

Let the measured apparatus operator be

$$Y(\hat{p}) = \sum_j p_{(2m+1)n_j} \vartheta_j(\hat{p}) \quad (4.9)$$

( $p_{(2m+1)n_j}$  being the central point of  $S_j$ ), or

$$Y = \sum_j j \vartheta_j(\hat{p}). \quad (4.10)$$

The equation (4.9) corresponds to inexact measurement of  $\hat{p}$ , the latter one means that the number  $j$  of range, to which  $p$  belongs, is measured. Note that we may set  $Y = \sum_j x_j \vartheta_j(\hat{p})$ , then we will have  $\langle [X \otimes I_A - I_S \otimes Y]^2 \rangle = 0$  as it follows from (5.5), (4.2), (4.8).

## 5 Selective collapse of the S-system state as a result of measuring apparatus variable $Y$

Now if we measure the physical quantity (4.7) or (4.8) and  $p$  proves to belong to  $\tilde{S}_l$ , the collapse

$$\rho(\varepsilon) \rightarrow \frac{1}{w'_l} [I_S \otimes \vartheta_l(\hat{p})] \rho(\varepsilon) [I_S \otimes \vartheta_l(\hat{p})] \quad (5.1)$$

(with  $w'_l = \text{Tr} [I_S \otimes \vartheta_l(\hat{p})] \rho(\varepsilon) [I_S \otimes \vartheta_l(\hat{p})]$ ) takes the form

$$\langle p_r | \rho(\varepsilon) | p_s \rangle \rightarrow \frac{1}{w'_l} \vartheta_l(p_r) \langle p_r | \rho(r) | p_s \rangle \vartheta_l(p_s) = \frac{1}{w'_l} E_l \rho_S E_l w_{s-(2m+1)n_l}^0 \delta_{rs}. \quad (5.2)$$

owing to (3.9), (4.8). In fact, applying (4.8) we have

$$\begin{aligned} \vartheta_l(p_r) w_{r-(2m+1)n_i}^0 \delta_{r-s-(2m+1)(n_i-n_j)} &= w_{r-(2m+1)n_i}^0 \delta_{r-s-(2m+1)(n_i-n_j)} \delta_{il} \\ &= w_{s-(2m+1)n_j}^0 \delta_{r-s-(2m+1)(n_i-n_j)} \delta_{il} \end{aligned}$$

and

$$w_{s-(2m+1)n_j}^0 \vartheta_l(p_s) = w_{s-(2m+1)n_l}^0 \delta_{jl}.$$

This leads to (5.2). Formula (5.2) means that the *a posteriori* state of quantum object S is  $E_l \rho_S E_l / w'_l = E_l \rho_S E_l / w_l$ .

However, the objection arises that it is incorrect to interpret the quantum collapse  $\rho_S \rightarrow E_l \rho_S E_l / w_l$  by another quantum collapse, namely by (5.1). In fact, matrix (3.9) does not commute with  $I_S \otimes \hat{p}$  and  $\{I_S \otimes Y\}$  and therefore consistency condition of the type (1.8) is violated. This condition would had been met for collapse

$$\rho(\varepsilon) \rightarrow \frac{1}{w'_l} \rho(\varepsilon) * \vartheta_l(\hat{p}), \quad (5.3)$$

but now (5.3) is not justified since it contradicts the collapse (5.1).

To overcome the above difficulty, the averaging with respect to some quantum or non-quantum variables should be done. There exist several lines of action and reasoning.

1. We suppose that non-quantum parameter  $\chi$  entering the right-hand side of (3.10) is random and uniformly distributed on the interval  $-\pi < \chi \leq \pi$ . Then averaging the right-hand side of (3.10) with respect to  $\chi$  leads to

$$\overline{\langle p_r | \rho(\varepsilon) | p_s \rangle} = \sum_j E_j \rho_S E_j w_{r-(2m+1)n_j}^0 \delta_{rs} \quad (5.4)$$

because the mean value of  $\exp[i(n_i - n_j)\chi]$  is  $\delta_{ij}$ . The matrix density (5.4) commutes with  $I_S \otimes \hat{p}$  and  $I_S \otimes Y(\hat{p})$ . Therefore the measurement of  $Y$  is classical-like (see Sect.1) and both the quantum collapse (5.1) and the classical one (5.3) may now be applied to (5.4). This gives the resulting *a posteriori* state  $E_l \rho_S E_l w_{r-(2m+1)n_l}^0 \delta_{rs} / w'_l$ . Averaging with respect to the apparatus parameter was used in [6] for explaining the non-selective collapse.

**2.** Another possibility is the averaging with respect to some quantum variables of the apparatus. We can restrict the operator algebra in which we are interested in. Let us only consider operator subalgebra  $\mathcal{A}_0$  generated by all operators of S-system (i.e. operators of the type  $D \otimes I_A$ ) and by operator  $I_S \otimes \hat{p}$ . The analogous type of the operator subalgebra (with coordinate taken instead of momentum) was considered in [1]. To be exact algebra of all operators commuting with  $Q = \kappa I \otimes \hat{q}$  was applied there for securing the consistancy condition by defining non-demolition observation continuous in time, the operator  $Y$  having both discrete and continuous spectrum. Earlier Araki [5] used a special subalgebra of operators for obtaining non-selective collapse in the limit  $t \rightarrow \infty$  for a particular choice of interaction.

The state functional (functional of mean values) for operators belonging to our subalgebra  $\mathcal{A}_0$  is defined with the help of operator (4.1):

$$\langle G \rangle = \sum_k \text{Tr}_S R(p_k) \langle p_k | G | p_k \rangle, \quad (5.5)$$

When we only consider operators from the subalgebra  $\mathcal{A}_0$  and use  $R(p_k)$ , the classical selective collapse

$$R(p_k) \rightarrow \frac{1}{w'_l} R(p_k) \vartheta_l(p_k) \quad (5.6)$$

analogous to transition to the conditional probability distribution takes place provided that the result of the measurement becomes known. According to (4.2), (4.8) this means transformation

$$R(p_k) \rightarrow \frac{1}{w'_l} E_l \rho_S E_l w_{k-(2m+1)n_l}^0$$

Summation with respect to apparatus momentum gives *a posteriori* state  $E_l \rho_S E_l / w'_l$  of the quantum object.

**3.** Suppose now that the quantum system interacts with two systems A and C, C being another copy of A-system considered earlier. Let it be in the

same initial state. Then A + C constitute a new complex apparatus. Averaging with respect to the C-system variables, i. e. considering subalgebra  $\mathcal{A}_0$  operators of the type  $D \otimes I_C$  ( $D$  being an operator on  $\mathcal{H}_S \otimes \mathcal{H}_A$ ) will solve the problem. For operators  $\tilde{D} = D \otimes I_C$  from  $\mathcal{A}_0$  the functional of mean values is  $\langle \tilde{D} \rangle = \text{Tr}_{S+A} D \rho_{S+A}$  with  $\rho_{S+A} = \text{Tr}_C \rho$ .

Now the total Hamiltonian takes the form

$$H(t) = H_S'' + H_A'' + H_C'' - \gamma B''(\hat{q}'' + Q'')\delta(t),$$

where  $H_S'' = H_S \otimes I_A \otimes I_C$ ,  $B'' = B \otimes I_A \otimes I_C$ ,  $\hat{q}'' = I_S \otimes \hat{q} \otimes I_C = I_S \otimes \hat{q}'$ ,  $Q'' = I_S \otimes I_A \otimes Q = I_S \otimes Q'$  and so on,  $Q$  being the coordinate of C, i.e. the operator on  $\mathcal{H}_C$ . Naturally the matrix

$$\rho = \rho_S \otimes \rho_A \otimes \rho_C$$

serves as the initial density matrix. In this case we have

$$\rho(\varepsilon) = \sum_{ij} E_i \rho_S E_j \otimes \exp \left[ \frac{i}{\hbar} \gamma b_i (\hat{q}' + Q') \right] (\rho_A \otimes \rho_C) \exp \left[ -\frac{i}{\hbar} \gamma b_j (\hat{q}' + Q') \right] \quad (5.7)$$

instead of (3.9). Since  $\hat{q}'$  commutes with  $Q'$  and  $I_A \otimes \rho_C$ , and  $Q'$  commutes with  $\rho_A \otimes I_C$ , this formula can be written as

$$\rho(\varepsilon) = \sum_{ij} E_i \rho_S E_j \otimes e^{i\gamma b_i \hat{q}'/\hbar} \rho_A e^{-i\gamma b_j \hat{q}'/\hbar} \otimes e^{i\gamma b_i Q'/\hbar} \rho_C e^{-i\gamma b_j Q'/\hbar}. \quad (5.8)$$

If we write the matrices  $r_{ij} = \exp(i\gamma b_i Q'/\hbar) \rho_C \exp(-i\gamma b_j Q'/\hbar)$  in the coordinate representation, after using (1.3) we have

$$r_{ij}(Q', Q) = \exp[i\hbar^{-1} \gamma a (n_i Q' - n_j Q)] \rho_C(Q', Q), \quad (5.9)$$

where

$$\rho_C(Q', Q) = L^{-1} \sum_{k=-m}^m \exp[2\pi i (Q' - Q)k/L] w_k^0 \quad (5.10)$$

((5.10) is analogous to (2.3)). From (5.9), (5.10) we see that setting  $\gamma a = 2\pi\hbar N/L$  ( $N$  is an integer) and taking the partial trace  $\text{Tr}_C$  with respect to C-system (i.e. integrating with respect to  $Q' = Q$ ) will give

$$\text{Tr}_C r_{ij} = \delta_{ij}.$$

Therefore we get from (5.8)

$$\text{Tr}_C \rho(\varepsilon) = \sum_j E_j \rho_S E_j \otimes \exp(i\gamma a n_j \hat{q}/\hbar) \rho_A \exp(-i\gamma a n_j \hat{q}/\hbar)$$

and

$$\langle p_k | \text{Tr}_C \rho(\varepsilon) | p_l \rangle = \sum_j E_j \rho_S E_j w_{k-(2m+1)n_j}^0 \delta_{kl}$$

for  $N = 2m + 1$ . Thus averaging with respect to all C-system quantum variables gives the same result as averaging in non-quantum random parameter  $\chi$ .

In the first and the third ways of reasoning we have obtained the *a posteriori* combined system state  $\tilde{\rho}_l^{(S)} \otimes \tilde{\rho}_l^{(A)}$ , where  $\tilde{\rho}_l^{(S)} = E_l \rho_S E_l / w'_l$ ,  $\tilde{\rho}_l^{(A)} = \sum_k |p_k\rangle w_{k-(2m+1)n_l}^0 \langle p_k|$ . This means that the quantum object is in the state  $E_l \rho_S E_l / w'_l$ . Due to normalization condition  $w'_l$  coincides with the probability  $w_l = \text{Tr}_S E_l \rho_S$  entering (1.1). So the transformation (1.1) of the object state takes place.

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## Appendix. Violation of the Heisenberg uncertainty relation?

The Heisenberg uncertainty relation  $\sigma_q \sigma_p \geq \hbar/2$  is the consequence of the well-known operator inequality

$$4\langle A^2 \rangle \langle B^2 \rangle \geq \left( i\langle [A, B] \rangle \right)^2 \quad (\text{A.1})$$

valid for any self-adjoint operators  $A$  and  $B$ . Of course, it should be valid in our case.

Let us map our coordinate space onto real axis in such a way that all points  $x_n = x_0 + nL$  are images of the same coordinate-space point. Here

$n$  is an arbitrary interger. All functions on the coordinate space should appear as periodical functions on the real axis. The momentum operator  $p = -i\hbar\partial/\partial x$  generates shifts

$$\exp(icp)\varphi(x) = \exp(c\hbar\partial/\partial x)\varphi(x) = \varphi(x + \hbar c)$$

in the real axis and coordinate space. The normalized eigenfunctions of  $p$  have the form

$$\varphi_k(x) = L^{-1/2} \exp(ip_k x/\hbar). \quad (\text{A.2})$$

They correspond to eigenvalues  $p_k = 2\pi\hbar k/L$ .

Now the question arises how to define the function  $q(M)$  in the coordinate space ( $M$  is its point), or, which is equivalent, the function  $q(x)$ . We cannot set  $q(x) = x$  since  $q(x)$  should be periodic. However, we should define  $q(x)$  in such a way that formula

$$\psi_k(q) := L^{-1/2} \exp[ip_k x(q)/\hbar] = L^{-1/2} \exp[ip_k q/\hbar],$$

which is analogous to (A.2), be valid. For this to be so  $q(x)$  should only differ from  $x$  by periodic jumps of magnitude  $\Delta x$  multiple to  $L$  at some points  $c_n = c_0 + nL$ . If  $0 < c \leq L/2$ , we may set

$$q(x) = x - L\eta(x - c) \quad \text{at} \quad -L/2 < x \leq L/2 \quad (\text{A.3})$$

with  $\eta(\xi) = (1 + \text{sign } \xi)/2$ . For function (A.3) and  $\hat{p} = -i\hbar\partial/\partial x$  we get

$$[p, q] = -i\hbar + i\hbar L\delta(x - c) \quad \text{at} \quad -L/2 < x \leq L/2. \quad (\text{A.4})$$

Averaging (A.4) or, to be exact, the matrix

$$[p, q]_{xx'} = -i\hbar[1 - L\delta(x - c)]\delta(x - x') \quad (\text{A.5})$$

with density matrix  $\rho_{x'x}$  of the type (2.3) we obtain  $\langle i[p, q] \rangle = 0$ . Therefore inequality (A.1) for  $A = q$ ,  $B = p$  gives  $\sigma_q \sigma_p \geq 0$ . So the Heisenberg uncertainty relations may be violated in our case.

Operator (A.5) in the momentum representation is of the form

$$\langle p_k | [p, q] | p_l \rangle = -i\hbar\delta_{kl} + i\hbar(-1)^{k-l} \quad (\text{A.6})$$

in the limit  $c \rightarrow L/2$ . Therefore  $\langle p_k | [p, q] | p_k \rangle = 0$ .

Note that the unusual commutativity relation (A.5), (A.6) leads to unusual dynamic equations. For example, in the case of an isolated apparatus with simple Hamiltonian  $H_A = p^2/(2m_0)$  the usual equation  $\dot{q} = p/m_0$  is not valid.

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